REVISTA DE LA
UNIÓN MATEMÁTICA ARGENTINA
Volumen 49, Número 2, 2008, Páginas 81–96

BEST LOCAL APPROXIMATIONS BY ABSTRACT NORMS WITH NON-HOMOGENEOUS DILATIONS

NORMA YANZÓN AND FELIPE ZÓ

In memorian of Mischa Cotlar.

ABSTRACT. We introduce a concept of best local approximation using abstract norms and non-homogeneous dilations. The asymptotic behavior of the normalized error function as well as the limit of some net of best approximation polynomials P_ε as $\varepsilon \to 0$ are studied.

1. INTRODUCTION.

The notion of a best local approximation of a function has been introduced by Chui, Shisha and Smith [6] in the seventies although its origin goes as far as the paper of J. Walsh [23]. A rather general view of the problem is as follows. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function in a normed space X with norm $\|.\|$. Let V denote a subset of X, consider k points $x_1, ..., x_k$ in $\mathbb{R}^n$ and small neighborhoods $V_\varepsilon(x_i)$ around each point x_i such that $V_\varepsilon(x_i)$ shrinks down to the point x_i as $\varepsilon \to 0$, for $i = 1, ..., k$. We wish to approximate f near the points $x_1, ..., x_k$ using an elements of V. For each $\varepsilon > 0$ we select a $P_\varepsilon(f) = P_\varepsilon \in V$ which minimizes

$$\|(f - P)\mathcal{X}_{V_\varepsilon}\|, \tag{1}$$

where $P \in V$ and $V_\varepsilon = \bigcup_{i=1}^k V_\varepsilon(x_i)$. If P_ε converges as $\varepsilon \to 0$ to an element $P_0(f) \in V$ then $P_0(f)$ is said to be a best local approximant of f at the points $x_1, ..., x_k$.

Thus we have $P_0(f)$ to be the set of cluster points of the net $\{P_\varepsilon(f)\}$ as $\varepsilon \to 0$, which may be the empty set, a singleton or a set with more than one element, see [7] for one dimensional examples with non smooth functions and [14] for n-dimensional examples where $P_0(f) = \emptyset$ or $\mathrm{card} P_0(f) > 1$ even for functions $f \in C^\infty$, the algebraic polynomials of degree at most m as the approximant class and $\|.\|$ to be

2000 *Mathematics Subject Classification.* 41A65.
Key words and phrases. best local approximations, function norms, non-homogeneous dilations.
The authors where supported by UNSL, CONICET and FONCYT grants.

the the L^2 norm. In many situations $P_0(f)$ has one element and it is called the best local approximation of f [23], [6], [7],[5], [25] and more recently [9], [10] for one point. The case of more than one point, sometimes called the best multipoint local approximation is fully treated in [4] where the L^p norm is used, see [20] and [1] for other approaches to best multipoint local approximations with L^p norms, for Orlicz norms see [13],[9],[15] and for a general family of norms [10] and [11].

The minimizing problem in (1) for the particular case $-1 \leq x_i \leq 1$ $V_\varepsilon(x_i) = (x_i - \varepsilon, x_i + \varepsilon)$ and $\|f\| = (\int_{-1}^{1} |f(x)|^p dx)^{\frac{1}{p}}$, $1 < p < \infty$ and $V = \pi^m$ the algebraic polynomials of degree at most m, is related to the following problem. For simplicity let us take $k = 1, x_1 = 0$ and let $\overline{P}(f)$ be the unique polynomial in π^m, which minimizes

$$\|(f - P)\|, \tag{2}$$

where $P \in \pi^m$. It is readily seen that $\overline{P}(f^\varepsilon)(\frac{x}{\varepsilon}) = P_\varepsilon(f)(x)$ where $f^\varepsilon(x) = f(\varepsilon x)$, and $P_\varepsilon(f)$ is the minimum problem described in (1) for this particular choice of norm. Not always the relationship between the best approximations $P_\varepsilon(f)$ and $\overline{P}(f)$ is so easily described as above, and some normalization in the norm used in problem (1) it is necessary to obtain a relationship between them see [18], [26] and [9]. Of course the problem (1) and the normalized one may have different solutions, see the last section of this paper. In this paper as it was done in [11] we study best approximation problems related to (2) and in term of these best approximations we define best approximations $P_\varepsilon(f)$ which play the role as the solutions of the problem (1) although in general they will give origin to different notions of best local approximations.

In [11] we studied the best local approximation problem,where the notion of closeness was given by a very general family of function seminorms acting on vector valued Lebesgue measurable functions. These seminorms embraced by far the norms used in these sort of problems, for example L^p, Orlicz or Lorentz norms. The fact to consider best local approximation problems on vector valued functions of several variables it was due to understand better the solution to the so called multipoint best local approximation problems given in [4], [13] and [20], also this general set up gives origin to best local approximation problems not considered before, even using the standard L^p norms.

The main goal of this paper is to consider, within the general frame given by [11], best local approximation on regions induced by dilations of the form $\delta_\varepsilon x = (\varepsilon^{\alpha_1} x_1, ..., \varepsilon^{\alpha_n} x_n)$ as treated in [26], [14].

However, we should point out that our presentation doves not cover all the problems of the paper [4], for example the case when we approach a function on small neighborhood $V_\varepsilon(x_1), ..., V_\varepsilon(x_k)$ with polynomials of degree at most n, the number k does not divide $n + 1$, and the neighborhood $V_\varepsilon(x_i)$ shrink dawn to the point x_i with different velocity at each x_i, for $i = 1, ..., k$.This last problem it was solved rather exhaustively for L^p norms in [4] but it remains open in other general norms, for example in Orlicz norms, for a recent contribution in this direction see [15].

It is known, [14], that when non-homogeneous dilations $(\varepsilon^{\alpha_1}x_1, ..., \varepsilon^{\alpha_n}x_n)$ are used in best local approximation problems, the class Π^m of algebraic polynomials of degree at most m is not suitable as an approximation class and should be replaced by a class $\Pi^{m,\alpha}$ which depends m de n-tuple $\alpha = (\alpha_1, ..., \alpha_n)$, see Definition 3.1.This paper extends results of [9] and [26] among others.

2. **The norm set up.**

We will work with a family of function seminorms $\|.\|_\varepsilon$, $0 \le \varepsilon \le 1$, acting on Lebesgue measurable functions $F : B \subset \mathbb{R}^n \to \mathbb{R}^k$, where $B = \{x \in \mathbb{R}^n : |x| \le 1\}$, and $|\ .\ |$ denotes the euclidean norm on $\mathbb{R}^n$.

We assume the following properties for the family of function seminorms $\|.\|_\varepsilon$, $0 \le \varepsilon \le 1$.

(1). For $F = (f_1, ...f_k)$, and $G = (g_1, ..., g_k)$, we have $\|F\|_\varepsilon \le \|G\|_\varepsilon$ for every $\varepsilon > 0$, provided $|f_i(x)| \le |g_i(x)|$, $i = 1, ..., k$, and $x \in B$.

(2). If 1 is the function $F(x) = (1, ..., 1)$, we have $\|1\|_\varepsilon < \infty$, for all $\varepsilon \ge 0$.

(3). For every $F \in C_k(B)$, we have $\|F\|_\varepsilon \to \|F\|_0$, as $\varepsilon \to 0$, where $C_k(B)$ is the set of continuous functions $F : B \subset \mathbb{R}^n \to \mathbb{R}^k$. Moreover $\|F\|_0$ is a norm on $C_k(B)$. From now on, if we do not specify the contrary, the statements will be valid for an abstract family of seminorms $\|.\|_\varepsilon$, $0 \le \varepsilon \le 1$, fulfilling conditions (1)-(3).

In order to give examples of norms $\|.\|_\varepsilon$, $0 \le \varepsilon \le 1$ with the properties (1)-(3) we recall a definition of convergence of measures early given in [16]. See also [2] for the notion of weak convergence of measures in general.

Definition 2.1. *Let μ_ε , $0 \le \varepsilon \le 1$, be a family of probability measures on B. We say that the measures μ_ε converge weakly in the proper sense to the measure μ_0 if we have*

$$\int_B f(x)\, d\mu_\varepsilon(x) \to \int_B f(x)\, d\mu_0(x), \quad f \in C_1(B),$$

and $\mu_0(B') > 0$ for any ball $B' \subseteq B$.

The assumption on the measure μ_0 implies that

$$\|F\|_\varepsilon = \|F\|_{L^p(\mu_\varepsilon)} = \left(\int_B \|F\|^p d\mu_\varepsilon\right)^{1/p},$$

is actually a norm on $C_k(B)$ for $\varepsilon = 0$ and $1 \leq p < \infty$, where $\| \ . \ \|$ stands for any monotone norm on $\mathbb{R}^k$. A seminorm fulfilling a property like (1) is called a monotone norm. We use a monotone norm on $\mathbb{R}^k$ to assure property (1) for the family of seminorms $\|.\|_\varepsilon$, $0 \leq \varepsilon \leq 1$. It is worthy to note we will not need this property on $\| \ . \ \|$ in proving some convergence results, see [10]

Let F be in $C_k(B)$; it is readily seen, by using the definition of weak convergence of measures, that there exists $\varepsilon_0 = \varepsilon_0(F) > 0$ such that if $\|F\|_\varepsilon = \|F\|_{L^p(\mu_\varepsilon)} = 0$, for some $0 < \varepsilon \leq \varepsilon_0$ then $F = 0$. Moreover we have that $\|F\|_\varepsilon = \|F\|_{L^p(\mu_\varepsilon)}$ converges as $\varepsilon \to 0$ to the norm $\|F\|_0 = \|F\|_{L^p(\mu_0)}$ if $F \in C_k(B)$.

In the next example, and for the most of the paper, we will consider a fixed notion of dilation on $\mathbb{R}^n$, namely

$$\delta_\varepsilon x = \delta_\varepsilon^\alpha x = (\varepsilon^{\alpha_1} x_1, ..., \varepsilon^{\alpha_n} x_n) \;\; \varepsilon > 0,$$

where $\alpha = (\alpha_1, ..., \alpha_n)$ are given positive numbers. Associated with the above (nonisotropic) dilation is the metric $r(x-y)$ on $\mathbb{R}^n$, where $r(x)$ is defined for any $x \neq 0$ as the unique positive number r which is the solution of $|\delta_{r^{-1}}(x)| = 1$, that is,

$$\sum_{j=1}^{n} \frac{x_j^2}{r^{2\alpha_j}} = 1 \quad .$$

The function $r(\ .\ )$ has the homogeneity property $r(\delta_\varepsilon x) = \varepsilon r(x)$ and there is polar like decomposition of $\mathbb{R}^n$ relative to r, i.e. to each vector $x \neq 0$ we assign $(r(x), x')$ where $\delta_r x' = x$, or $x' = \delta_{r^{-1}}(x)$ is in the unit sphere S^{n-1}. Finally we can integrate in polar like coordinates, according to the next formula

$$\int_{\mathbb{R}^n} f(x)dx = \int_0^\infty r^{|\alpha|-1} (\int_{S^{n-1}} f(\delta_r x')(Px', x')dx') \; dr, \tag{3}$$

where dx' denotes the Lebesgues measure over the unit sphere S^{n-1}, $|\alpha| = \Sigma_{i=1}^n \alpha_i$ and P is the diagonal matrix which generates the semigroup of dilations $\delta_\varepsilon = \exp P \ln \varepsilon$. We refer to the article [8] for an early use of this dilations in harmonic analysis or for more general dilations see [21]. In the following example we introduce measure μ_ε adapted to the dilations $\delta_\varepsilon x$ which plays an analogous role to those introduced in [18].

Example 2.2. *Let $\alpha = (\alpha_1, ..., \alpha_n)$ be a fix n-tuple of real numbers such that $\alpha_i \geq 1$, the measures μ_ε are given by*

$$\mu_\varepsilon(E) = \int_E \varepsilon^{|\alpha|} w\left(\delta_\varepsilon t\right) W^{-1}\left(\varepsilon\right) dt, \tag{4}$$

where , $\delta_\varepsilon t = (\varepsilon^{\alpha_1} t_1, ..., \varepsilon^{\alpha_n} t_n)$, $B_\varepsilon = \{\delta_\varepsilon t : \; t \in B\}$ and $W(\varepsilon) = \int_{B_\varepsilon} w(t)\, dt$. The following condition on the weight function w will be assumed

$$W(\varepsilon) = A\varepsilon^{\beta+|\alpha|}\left(1 + o\left(1\right)\right), as \; \varepsilon \to 0, \; A > 0, \; \beta + |\alpha| > 0. \tag{5}$$

We say that the weight function w is a radial function with respect to r if $w(x) = w(y)$ when $r(x) = r(y)$.

Remark 2.3. *If the weight function w is a radial function, then the measures $\mu_\varepsilon(E)$ introduced in Example 2.2 converges weakly to the measure*

$$\mu_0(E) = C(|\alpha| + \beta) \int_E r^\beta(x)dx, \tag{6}$$

where C is a constant depending upon the weight function w.

The proof of this last remark follows the same pattern of Lemma 1 in [18] and now we will point out the necessary modifications of the proof.

Let us consider the function $w_\varepsilon(t) = \varepsilon^{|\alpha|}W^{-1}(\varepsilon)w(\delta_\varepsilon t)$ and set $\|Q\|^p_{(p,w_\varepsilon)} = \varepsilon^{|\alpha|}W^{-1}(\varepsilon)\int_B |Q(t)|^p w(\delta_\varepsilon t)dt$ for $Q \in \pi^m$.

To deal with the weights w_ε we consider a change of variables of polar type induced by the function r given by (3) .

Using the above formula and following the steps of the proof of Lemma 1 in [18], we can see that

$$|\|Q\|^p_{(p,w_\varepsilon)} - \|Q\|^p_{(p,\overline{w})}| \leq o(1)\|Q\|^p_{(p,\overline{w})}, \tag{7}$$

where $\overline{w}(t) = w_n^{-1}(\beta + |\alpha|)r(t)^\beta$ and $w_n^{-1} = \int_{S^{n-1}}(Px', x')dx'$. Now it is easy to obtain the weak convergence of the measures μ_ε to μ_0 and that the constant C in equation (6) it turns to be w_n^{-1}.

3. The Taylor polynomial and the limit of best approximation polynomials.

Throughout this paper $\alpha = (\alpha_1, ..., \alpha_n)$ will denote a fix n-tuple of real numbers such that $\alpha_i \geq 1$ and $\min \alpha_i = 1$.

Definition 3.1. ***The class*** $\pi^{\mathbf{m},\alpha}$. *Given a positive number m we say that a real polynomial p is in the class $\pi^{m,\alpha}$ if it is of the form*

$$p(x) = \sum_{\alpha.\beta \leq m} a_\beta x^\beta = \sum_{\alpha.\beta \leq m} a_\beta x_1^{\beta_1}...x_n^{\beta_n},$$

where each $\beta \in \mathbb{N}^n$ and at least one of them satisfies $\alpha.\beta = m$. A polynomial p is of α-degree m if $p \in \pi^{m,\alpha}$

Note that in the case $\alpha = (1, ..., 1)$ we obtain the classical definition of polynomial of degree m. We denote by $\Pi_k^{m,\alpha}$ the set $\{P = (p_1, ..., p_k) : p_i \in \pi^{m,\alpha}\}$.

It is worthy to note that when we make reference to a polynomial of α-degree m we do not mean the classic degree of polynomial. For example, for $\alpha = (1, 3/2)$ there exist polynomials of α-degree $3/2$ and they are of the form $ax^{(0,0)} + bx^{(1,0)} + cx^{(0,1)}$, for a, b, c in $\mathbb{R}$.

Given a function $F : B \subset \mathbb{R}^n \to \mathbb{R}^k$, and a family of seminorms $\|.\|_\varepsilon$, $0 \leq \varepsilon \leq 1$, as in section 2, we introduce a general version of the "Peano's definition" of the Taylor polynomial, see [8, 26, 17, 18, 22]. We will use the notation $F^\varepsilon(x) = F(\delta_\varepsilon x) = F(\varepsilon^{\alpha_1} x_1, ..., \varepsilon^{\alpha_n} x_n)$.

Definition 3.2. *A function $F : B \subset \mathbb{R}^n \to \mathbb{R}^k$, has a Taylor Polynomial of α-degree m, if there exists $T_{m,\alpha} = T_{m,\alpha}(F) \in \Pi_k^{m,\alpha}$ such that*

$$\|F^\varepsilon - T_{m,\alpha}^\varepsilon\|_\varepsilon = o(\varepsilon^m), \quad \varepsilon \to 0.$$

We write $F \in t^{m,\alpha}$ if the function F has a Taylor Polynomial of α-degree m.

To prove the uniqueness of the Taylor polynomial we need the next result which is a consequence of an usual compactness argument.The proof is essentially given in [11], and it depends basically of the properties of the family of seminorms. See also [18].

Proposition 3.3. *There exist $C = C(m, k, \alpha)$ and $0 < \varepsilon(m, k, \alpha)$ such that for every $0 < \varepsilon \leq \varepsilon(m, k, \alpha)$,*

$$C^{-1}\|P\|_0 \leq \|P\|_\varepsilon \leq C\|P\|_0,$$

for every $P \in \Pi_k^{m,\alpha}$.

We make use of the standard notation, $\partial^\gamma f = \frac{\partial^{\gamma_1}}{\partial x_1^{\gamma_1}} ... \frac{\partial^{\gamma_n}}{\partial x_n^{\gamma_n}}(f)$ for $\gamma \in \mathbb{N}^n$.

Proposition 3.4. *There exists a constant $C > 0$, depending only on m, k, n, α and the family of seminorms $\| \, . \, \|_\varepsilon$ such that for any $P = (p_1, ..., p_k) \in \Pi_k^{m,\alpha}$ there holds*

$$|\partial^\alpha p_i(0)| \leq C\varepsilon^{-\alpha.\gamma}\|P^\varepsilon\|_\varepsilon,$$

for any $0 \leq \alpha.\gamma \leq m$ and $i = 1, ..., k$.

Proof. We consider the seminorm $\|P\| = \|\partial^\gamma P(x)\| = \max_i \|\partial^\gamma p_i(x)\|$. Using the above proposition we have

$$|\partial^\gamma p_i^\varepsilon(x)| \leq C\|P^\varepsilon\|_\varepsilon,$$

for a constant $C > 0$. Since $\partial^\gamma p_i^\varepsilon(0) = \varepsilon^{\alpha\cdot\gamma}\partial^\gamma p_i(0)$, the proposition follows. □

The next proposition is a consequence of Proposition 3.4.

Proposition 3.5. *The polynomial* $T_{m,\alpha} = T_{m,\alpha}(F) \in \Pi_k^{m,\alpha}$ *in Definition 3.2 is unique.*

Proposition 3.6. *If the function* F *has the Taylor polynomial of* α*-degree* m, $T_{m,\alpha}(x) = \sum_{0\le\alpha.\beta\le m} A_\beta x^\beta$, *then the Taylor polynomial of* α*-degree* $l \le m$ *is given by* $T_{l,\alpha}(x) = \sum_{0\le\alpha.\beta\le l} A_\beta x^\beta$ *(if there exists* β *such that* $\alpha.\beta = l$ *) .We set* $\partial^\beta F(0)$ *for the vector* $\beta! A_\beta$*.*

Proof. We has

$$\|F^\varepsilon - T^\varepsilon_{l,\alpha}\|_\varepsilon \le o(\varepsilon^m) + \varepsilon^s \|P\|_\varepsilon = o(\varepsilon^l),$$

where $s = min\{\alpha.\beta : l < \alpha.\beta \le m\}$ and $P \in \Pi_k^{m,\alpha}$. □

Let $\|F\|_\varepsilon$, $0 \le \varepsilon \le 1$, be a family of seminorms and $F : B \subset \mathbb{R}^n \to \mathbb{R}^k$, be a fixed measurable function such that $\|F\|_\varepsilon$ and $\|F^\varepsilon\|_\varepsilon$ are finite for all ε. For any such F has a meaning the following definition.

Definition 3.7. *Set* $\|F\|^*_\varepsilon = \|F^\varepsilon\|_\varepsilon$, *and* $P_{\varepsilon,\alpha} = P_{\varepsilon,\alpha}(F)$ *for any polynomial in* $\Pi_k^{m,\alpha}$ *which minimizes* $\|F - P\|^*_\varepsilon$, $P \in \Pi_k^{m,\alpha}$.

Although the best approximation polynomial $P_{\varepsilon,\alpha}(F)$ is not unique in general, through this paper the notation $P_{\varepsilon,\alpha}(F)$ does not mean a set of best approximation polynomials but any arbitrarily chosen polynomial in this set. We have the existence of $P_{\varepsilon,\alpha}(F)$, at least for all small ε, by Proposition 3.3.

The next statement has its origin in [23] using the L^∞ norm, and since then similar versions in L^p in one and several real variables appeared. Results dealing with weighted Luxemburg norms appeared recently in [9] and [10].

Theorem 3.8. *If* $F \in t^{m,\alpha}$, *then* $P_{\varepsilon,\alpha} \to T_{m,\alpha}(F)$ *as* $\varepsilon \to 0$.

Proof. In fact $\|P^\varepsilon_{\varepsilon,\alpha} - T^\varepsilon_{m,\alpha}(F)\|_\varepsilon \le 2\|F^\varepsilon - T^\varepsilon_{m,\alpha}(F)\|_\varepsilon = o(\varepsilon^m)$, and by Proposition 3.4 it follows $|[P^{(\beta)}_{\varepsilon,\alpha}]_i(0) - [T^{(\beta)}_{m,\alpha}(F)]_i(0)| \le \varepsilon^{-\alpha.\beta} C \|P^\varepsilon_{\varepsilon,\alpha} - T^\varepsilon_{m,\alpha}(F)\|_\varepsilon$, and $0 \le \alpha.\beta \le m$. □

As in [6] we call the limit of $P_{\varepsilon,\alpha}(F)$ as $\varepsilon \to 0$, the best local approximation to F .

4. The asymptotic behavior of the error.

Let $\mathcal{A}$ be a subspace of polynomials $\Pi_k^{m,\alpha} \subseteq \mathcal{A} \subseteq \Pi_k^{l,\alpha}$ and let $F : B \subset \mathbf{R}^n \to \mathbb{R}^k$, be a Lebesgue measurable function. Set $P_{\varepsilon,\alpha} \in \mathcal{A}$ for a polynomial which is a best approximation of the function F with the seminorm $\|F\|^*_\varepsilon = \|F^\varepsilon\|_\varepsilon$. Observe that $P^\varepsilon_{\varepsilon,\alpha}$ is a polynomial in $\mathcal{A}^\varepsilon = \{P^\varepsilon : P \in \mathcal{A}\}$ which is a best approximation of the function F^ε with the seminorm $\| \,.\, \|_\varepsilon$ from the class $\mathcal{A}^\varepsilon$, and we will also denote it by $P_{\mathcal{A}^\varepsilon,\varepsilon,\alpha}(F^\varepsilon)$. We insist that $P_{\mathcal{A}^\varepsilon,\varepsilon,\alpha}(F^\varepsilon)$ means, in our notation, a fixed best approximation polynomial and not a set of them.

Let $E_\varepsilon(F)$ be the error function $\varepsilon^{-\overline{m}}(F^\varepsilon - P^\varepsilon_{\varepsilon,\alpha})$ where $\overline{m} = min\{h \in (m,\infty) : \exists\beta \;\; with\; \alpha.\beta = h\}$. Next, we will obtain an expression for the function $E_\varepsilon(F)$ which has its origin in [19] see also [17, 18].

Let F be in $t^{(\overline{m},\alpha)}$ and set $T_{\overline{m},\alpha}$ for the Taylor polynomial of F of α-degree $\overline{m}$; then by definition we have $F^\varepsilon = T^\varepsilon_{\overline{m},\alpha} + \varepsilon^{\overline{m}} R^\varepsilon_{\overline{m},\alpha}$, with $\|R^\varepsilon_{\overline{m},\alpha}\|_\varepsilon = o(1)$, and $R_{\overline{m},\alpha}(x) = \varepsilon^{-\overline{m}}(F(x) - T_{\overline{m},\alpha}(x))$. Moreover, observe that $\lambda P_{\mathcal{A}^\varepsilon,\varepsilon,\alpha}(F^\varepsilon) = P_{\mathcal{A}^\varepsilon,\varepsilon,\alpha}(\lambda F^\varepsilon)$ and $T^\varepsilon_{m,\alpha} + P_{\mathcal{A}^\varepsilon,\varepsilon,\alpha}(F^\varepsilon) = P_{\mathcal{A}^\varepsilon,\varepsilon,\alpha}((T_{m,\alpha} + F)^\varepsilon)$, where we have used that $T_{m,\alpha} \in \mathcal{A}$. Using the equality $\Phi_{\overline{m},\alpha} + R^\varepsilon_{\overline{m},\alpha} = \varepsilon^{\overline{m}} F^\varepsilon - \varepsilon^{\overline{m}} T_{m,\alpha}$ we obtain the following result

Proposition 4.1. *Let F be a function in $t^{\overline{m},\alpha}$, and $\Phi_{\overline{m},\alpha} = T_{\overline{m},\alpha} - T_{m,\alpha}$. Then*

$$E_\varepsilon(F) = \Phi_{\overline{m},\alpha} + R^\varepsilon_{\overline{m},\alpha} - P_{\mathcal{A}^\varepsilon,\varepsilon,\alpha}(\Phi_{\overline{m},\alpha} + R^\varepsilon_{\overline{m},\alpha}),$$

$\|R^\varepsilon_{\overline{m},\alpha}\|_\varepsilon = o(1)$, *as* $\varepsilon \to 0$.

Proposition 4.1 is useful when $\mathcal{A}^\varepsilon = \mathcal{A}$ for every $\varepsilon > 0$. The case $\mathcal{A} = \Pi^{m,\alpha}_k$ with $\alpha = (1, ..., 1)$ was considered in [18] and [17] for weighted L^p norms and in [9] for the Luxemburg norm. It is easy to find $\Pi^{m,\alpha}_k \subsetneq \mathcal{A} \subseteq \Pi^{l,\alpha}_k, m < l$, and $\mathcal{A}^\varepsilon = \mathcal{A}$ for every $\varepsilon > 0$. The following result is relevant to this matter.

Theorem 4.2. *Let $\mathcal{A}$ be a subspace of polynomials such that $\Pi^{m,\alpha}_k \subseteq \mathcal{A} \subseteq \Pi^{\overline{m},\alpha}_k$. Then $\mathcal{A}^\varepsilon = \mathcal{A}$ for all $\varepsilon > 0$.*

Proof. Let $\mathcal{A}_1 = \{P - T_{m,\alpha}(P) : P \in \mathcal{A}\}$. We shall see if $\mathcal{A}^\varepsilon_1 = \mathcal{A}_1$ for all $\varepsilon > 0$, then $\mathcal{A}^\varepsilon = \mathcal{A}$ for all $\varepsilon > 0$. Let H be in $\mathcal{A}^\varepsilon$, i.e., $H = Q^\varepsilon$ with $Q \in \mathcal{A}$. As $Q - T_{m,\alpha}(Q) \in \mathcal{A}_1$ we have

$$Q^\varepsilon - T_{m,\alpha}(Q^\varepsilon) = Q^\varepsilon - (T_{m,\alpha}(Q))^\varepsilon \in \mathcal{A}^\varepsilon_1.$$

Thus $Q^\varepsilon - T_{m,\alpha}(Q^\varepsilon) \in \mathcal{A}_1$ and there exists $V \in \mathcal{A}$ such that $Q^\varepsilon - T_{m,\alpha}(Q^\varepsilon) = V - T_{m,\alpha}(V)$. Therefore $Q^\varepsilon - V \in \mathcal{A}$, so $H \in \mathcal{A}$. We have proved that $\mathcal{A}^\varepsilon \subset \mathcal{A}$ for all $\varepsilon > 0$. Since $\mathcal{A}^{\frac{1}{\varepsilon}} \subset \mathcal{A}$ we get $\mathcal{A} = (\mathcal{A}^{\frac{1}{\varepsilon}})^\varepsilon \subset \mathcal{A}^\varepsilon$. Therefore, $\mathcal{A}^\varepsilon = \mathcal{A}$ for all $\varepsilon > 0$.

Clearly, there is a linear space $W \subset \mathbb{R}^k$ such that $\mathcal{A}_1 = \{Ax^\beta : \alpha.\beta = \overline{m}\}$ and $\{A \in W\}$, then $\mathcal{A}^\varepsilon_1 = \mathcal{A}_1$. □

Theorem 4.3. *Let F be in $t^{\overline{m},\alpha}$, and $\mathcal{A}^\varepsilon = \mathcal{A}$ for every $\varepsilon > 0$. Then (a) $\|E_\varepsilon(F)\|_\varepsilon \to \|\Phi_{\overline{m},\alpha} - P_{\mathcal{A},0}(\Phi_{\overline{m},\alpha})\|_0$,*
as $\varepsilon \to 0$.

(b) $\|E_\varepsilon(F) - (\Phi_{\overline{m},\alpha} - P_{\mathcal{A},0}(\Phi_{\overline{m},\alpha}))\|_\varepsilon \to 0$,
as $\varepsilon \to 0$ *if* $\| \, \|_0$ *is a strictly convex norm.*
We have denoted by $P_{\mathcal{A},0}(\Phi_{\overline{m},\alpha})$ *a polynomial in* $\mathcal{A}$ *which is a best approximation of* $\Phi_{\overline{m},\alpha}$ *with respect to the norm* $\| \, . \|_0$.

Proof. Let us begin with (a). By Proposition 4.1 we have, for any $P \in \mathcal{A}$,

$$\|E_\varepsilon(F)\|_\varepsilon \leq \|\Phi_{\overline{m},\alpha} + R^\varepsilon_{\overline{m},\alpha} - P\|_\varepsilon = \|\Phi_{\overline{m},\alpha} - P\|_\varepsilon + o(1) = \|\Phi_{\overline{m},\alpha} - P\|_0 + o(1),$$

as $\varepsilon \to 0$. Therefore

$$\overline{\lim_{\varepsilon \to 0}} \|E_\varepsilon(F)\|_\varepsilon \leq \|\Phi_{\overline{m},\alpha} - P_{\mathcal{A},0}\|_0.$$

Let (ε_k) be a sequence tending to zero such that

$$\underline{\lim}_{\varepsilon \to 0} \|E_\varepsilon(F)\|_\varepsilon = \lim_{\varepsilon_k \to 0} \|E_{\varepsilon_k}(F)\|_{\varepsilon_k}.$$

Set $P_k = P_{\mathcal{A}^{\varepsilon_k},\varepsilon_k}(\Phi_{\overline{m},\alpha} + R^{\varepsilon_k}_{\overline{m},\alpha})$; then $\|P_k\|_{\varepsilon_k} \leq 2\|\Phi_{\overline{m},\alpha} + R^{\varepsilon_k}_{\overline{m},\alpha}\|_{\varepsilon_k} = 2\|\Phi_{\overline{m},\alpha}\|_{\varepsilon_k} + o(1)$. By Proposition 3.3 we can select a convergent subsequence of P_k which is again denoted by P_k and then we have $\|\Phi_{\overline{m},\alpha} - P_k\|_0 = \|\Phi_{\overline{m},\alpha} - P_k\|_{\varepsilon_k} + o(1)$, as $\varepsilon_k \to 0$. Then $\|\Phi_{\overline{m},\alpha} - P_{\mathcal{A},0}(\Phi_{\overline{m},\alpha})\|_0 \leq \|\Phi_{\overline{m},\alpha} - P_k\|_0 = \|\Phi_{\overline{m},\alpha} - P_k\|_{\varepsilon_k} + o(1) = \|\Phi_{\overline{m},\alpha} + R^{\varepsilon_k}_{\overline{m},\alpha} - P_k\|_{\varepsilon_k} + o(1)$. Thus we have

$$\|\Phi_{\overline{m},\alpha} - P_{\mathcal{A},0}(\Phi_{\overline{m},\alpha})\|_0 \leq \underline{\lim}_{\varepsilon_k \to 0} \|E_{\varepsilon_k}(F)\|_{\varepsilon_k}.$$

To prove (b), consider any sequence $\varepsilon_k \to 0$ and select $P_k = P_{\mathcal{A}^{\varepsilon_k},\varepsilon_k}(\Phi_{\overline{m},\alpha} + R^{\varepsilon_k}_{\overline{m},\alpha})$, then $\|E_{\varepsilon_k}(F)\|_{\varepsilon_k} = \|\Phi_{\overline{m},\alpha} + R^{\varepsilon_k}_{\overline{m},\alpha} - P_k\|_{\varepsilon_k}$. We will prove $P_k \to P_{\mathcal{A},0}(\Phi_{\overline{m},\alpha})$, which implies (b). In fact we may assume, by taking subsequences if it is necessary, that $P_k \to P_0 \in \mathcal{A}$, as $\varepsilon_k \to 0$. Thus by (a) $\|\Phi_{\overline{m},\alpha} - P_0\|_0 = \|\Phi_{\overline{m},\alpha} - P_{\mathcal{A},0}(\Phi_{\overline{m},\alpha})\|_0$. Since $\|.\|_0$ is a strictly convex norm we have $P_0 = P_{\mathcal{A},0}(\Phi_{\overline{m},\alpha})$. □

Consider the set $\mathcal{A} = \times_{i=1}^k \pi^{m_i}$, $\; m \leq m_i \; i = 1, ..., k$. where $\alpha.\beta = m_i$, for $\beta \in \mathbb{N}^n$. Then $\Pi_k^{m,\alpha} \subseteq \mathcal{A} \subseteq \Pi_k^{l,\alpha}$, and $\mathcal{A}^\varepsilon = \mathcal{A}$, for every $\varepsilon > 0$.

We now introduce an useful example of a subspace $\mathcal{A}$ such that $\mathcal{A}^\varepsilon \neq \mathcal{A}$. Consider the set

$$\mathcal{A}(l,k) = \mathcal{A}(l; x_1, ..., x_k) = \{Lp \;\; : p \in \pi^{l,\alpha}\}, \tag{8}$$

where, $Lp(s) = (p(x_1 + s), ..., p(x_k + s))$, and $-1 < x_1 < ... < x_k < 1$. Now $\mathcal{A} \bigcap \mathcal{A}^\varepsilon = \{(c, ..., c) : c \in \mathbb{R}\}$, see [11] in Proposition 4.2. Thus it is not possible to use Theorem 4.3 to study the function error with $\mathcal{A} = \mathcal{A}(l,k)$. The next condition on $\mathcal{A}$ will be significant in the future and it was used in [11] to consider cases such as $\mathcal{A}(l,k)$.

A subspace of polynomials which does not satisfies $\mathcal{A}^\varepsilon = \mathcal{A}$ is given in the following example.

Example 4.4. *We denote by $\mathcal{Q}_{\mathrm{V},\beta^1,\ldots,\beta^s}$ the set of all algebraic polynomials of the form*

$$a_1 x^{\beta^1} + \ldots + a_s x^{\beta^s},$$

where β^i, $i = 1, \ldots, s$ are fixed vectors in $\mathbb{N}^n$, $\beta^1.\alpha = \overline{m}$, $\beta^j.\alpha > \overline{m}$ for $j = 2, \ldots, s$ and $\beta^j.\alpha \neq \beta^k.\alpha$ for $j \neq k$. Moreover $(a_1, \ldots, a_s) = c(v_1, \ldots, v_s)$ whit $v_1 \neq 0$, $c \in \mathbb{R}$ and $V = (v_1, \ldots, v_s)$ is a fixed vector in $\mathbb{R}^s$.

Let $\mathcal{A} = \Pi_k^{m,\alpha} \bigoplus \mathcal{B}$, where $\mathcal{B} = \{(p_1, \ldots, p_k) : p_i \in \mathcal{Q}_{V_i,\beta^{1,i},\ldots,\beta^{s,i}}\}$. It is clear that $\mathcal{A}^\varepsilon \neq \mathcal{A}$.

Condition 4.5. *For $\Pi_k^{m,\alpha} \subseteq \mathcal{A} \subseteq \Pi_k^{l,\alpha}$, we assume that if $P \in \mathcal{A}$ and $T_{\overline{m},\alpha}(P) = 0$, then $P = 0$. Where $\overline{m} = min\{h \in \mathbb{R} : h > m\}$ and $\exists\beta$ with $\{\alpha.\beta = h\}$.*

Let $\mathcal{A} = \Pi_k^{m,\alpha} \bigoplus \mathcal{B}$ be as in Example 4.4 , then the Condition 4.5 holds. We consider again the error function $E_\varepsilon(F) = \varepsilon^{-\overline{m}}(F^\varepsilon - P_\varepsilon^\varepsilon)$, where $P_\varepsilon \in \mathcal{A}$ and $P_\varepsilon^\varepsilon = P_{\mathcal{A}^\varepsilon,\varepsilon}(F^\varepsilon)$. Set $G = F - T_{m,\alpha}(F)$ and recall that $T_{m,\alpha}(F) \in \mathcal{A}$. Then $E_\varepsilon(F) = E_\varepsilon(G)$. If $F \in t^{\overline{m},\alpha}$ we have

$$E_\varepsilon(F) = \Phi_{\overline{m},\alpha} - \varepsilon^{-\overline{m}} P_{\mathcal{A}^\varepsilon,\varepsilon}(G^\varepsilon) + o(1), \tag{9}$$

as $\varepsilon \to 0$, and $\Phi_{\overline{m},\alpha} = T_{\overline{m},\alpha}(F) - T_{m,\alpha}(F)$. The next theorem give us a useful expression for the error function $E_\varepsilon(F)$ as well as we know the polynomials $\{U_\varepsilon\}_{\varepsilon>0}$ and $\{P_\varepsilon\}_{\varepsilon>0}$ used to describe it. With the notation $\mathcal{A}_0 = \{P \in \mathcal{A} : T_{m,\alpha}(P) = 0\}$, observe that $\mathcal{A}$ is the direct sum $\Pi_k^{m,\alpha} \bigoplus \mathcal{A}_0$.

Theorem 4.6. *Let F be a function in $t^{\overline{m},\alpha}$, and assume Condition 4.5 for $\mathcal{A}$. Set $\overline{P}_\varepsilon^\varepsilon = P_{\mathcal{A}^\varepsilon,\varepsilon}((F - T_{m,\alpha}(F))^\varepsilon)$, with $\bar{P}_\varepsilon \in \mathcal{A}$, $U_\varepsilon = \bar{P}_\varepsilon - T_{m,\alpha}(\bar{P}_\varepsilon)$, and $V_\varepsilon = \varepsilon^{-\overline{m}} T_{m,\alpha}^\varepsilon(\overline{P}_\varepsilon)$. Then $U_\varepsilon \in \mathcal{A}_0$ and $V_\varepsilon \in \Pi_k^{m,\alpha}$ and*

$$E_\varepsilon(F) = \Phi_{\overline{m},\alpha} - T_{\overline{m},\alpha}(U_\varepsilon) - V_\varepsilon + o(1), \tag{10}$$

as $\varepsilon \to 0$. Moreover the two families of polynomials $\{U_\varepsilon\}_{\varepsilon>0}$ and $\{V_\varepsilon\}_{\varepsilon>0}$ are uniformly bounded in ε for a fixed norm $\| \,.\, \|$.

Proof. By (9) we have

$$E_\varepsilon(F) = \Phi_{\overline{m},\alpha} - \varepsilon^{-\overline{m}}({U_\varepsilon}^\varepsilon + {T_{m,\alpha}}^\varepsilon(\overline{P}_\varepsilon)) + o(1).$$

Or else, since $T_{m,\alpha}(U_\varepsilon) = 0$,

$$E_\varepsilon(F) = \Phi_{\overline{m},\alpha} - T_{\overline{m},\alpha}(U_\varepsilon) - \varepsilon^{-\overline{m}} {T_{m,\alpha}}^\varepsilon(\overline{P}_\varepsilon) + o(1),$$

as $\varepsilon \to 0$, which is (10).

Now we will prove $\{U_\varepsilon\}_{\varepsilon>0}$ and $\{V_\varepsilon\}_{\varepsilon>0}$ are uniformly bounded in ε. For a norm $\| \quad \|$ in $\mathbb{R}^k$, the expression

$$\max_{\alpha.\gamma\leq\overline{m}} \|\partial^\gamma P(0)\|,$$

is a norm on $\mathcal{A}$. Here we are using that the subspace $\mathcal{A}$ fulfills Condition 4.5.

Since $\overline{P}_\varepsilon^\varepsilon = P_{\mathcal{A}^\varepsilon,\varepsilon}(G^\varepsilon)$, with $G = F - T_{m,\alpha}(F)$, we have $\|\bar{P}_\varepsilon^\varepsilon\|_\varepsilon \leq 2\|G^\varepsilon\|_\varepsilon \leq 2\|T^\varepsilon_{\overline{m},\alpha}(G)\|_\varepsilon + o(\varepsilon^{\overline{m}}) = O(\varepsilon^{\overline{m}})$.
By Proposition 3.4 we have $\|\partial^\gamma \bar{P}_\varepsilon(0)\| = O(\varepsilon^{-\alpha.\gamma}\|\bar{P}_\varepsilon^\varepsilon\|_\varepsilon) = O(\varepsilon^{\overline{m}-\alpha.\gamma})$, for $\alpha.\gamma \leq \overline{m}$. Thus $\bar{P}_\varepsilon$ and hence $U_\varepsilon = \bar{P}_\varepsilon - T_{m,\alpha}(\bar{P}_\varepsilon)$ are uniformly bounded in $\varepsilon > 0$. To estimate the polynomials $V_\varepsilon = \varepsilon^{-(\overline{m})}T^\varepsilon_{m,\alpha}(\bar{P}_\varepsilon)$, we note that $\partial^\gamma V_\varepsilon(0) = \varepsilon^{-\overline{m}}\varepsilon^{\alpha.\gamma}\partial^\gamma \bar{P}_\varepsilon(0)$, for $\alpha.\gamma \leq m$. Then $\|\partial^\gamma V_\varepsilon(0)\| = O(1)$; recall that $V_\varepsilon \in \Pi_k^{m,\alpha}$, and $\max_{\alpha.\gamma\leq m}\|\partial^\gamma V_\varepsilon(0)\|$ is a norm there. □

Theorem 4.7. *Let F be in $t^{\overline{m},\alpha}$ and assume Condition 4.5 for $\mathcal{A}$. Then $\|E_\varepsilon(F)\|_\varepsilon$ tends to*

$$\min\{\|T_{\overline{m},\alpha}(G-U) - V\|_0 : U \in \mathcal{A}_0,\ V \in \Pi_k^{m,\alpha}\},$$

as $\varepsilon \to 0$, and $G = F - T_{m,\alpha}(F)$.

Proof. We will prove the following inequality

$$\overline{\lim_{\varepsilon\to 0}}\, \|E_\varepsilon(F)\|_\varepsilon \leq \inf_{U\in\mathcal{A}_0,\ V\in\Pi_k^{m,\alpha}} \|T_{\overline{m},\alpha}(G-U) - V\|_0 \leq \underline{\lim}_{\varepsilon\to 0}\, \|E_\varepsilon(F)\|_\varepsilon. \tag{11}$$

Let $U \in \mathcal{A}_0$ and $V \in \Pi_k^{m,\alpha}$ be two arbitrary polynomials and set $Z_\varepsilon \in \Pi_k^{m,\alpha}$ with $Z_\varepsilon^\varepsilon = \varepsilon^{\overline{m},\alpha}V$ and $U + Z_\varepsilon \in \mathcal{A}$. Then

$$\|E_\varepsilon(F)\|_\varepsilon = \|E_\varepsilon(G)\|_\varepsilon = \varepsilon^{-\overline{m}}\|G^\varepsilon - \bar{P}_{\mathcal{A}^\varepsilon,\varepsilon}(G^\varepsilon)\|_\varepsilon$$

$$\leq \varepsilon^{-\overline{m}}\|G^\varepsilon - (U+Z_\varepsilon)^\varepsilon\|_\varepsilon = \varepsilon^{-\overline{m}}\|T^\varepsilon_{\overline{m},\alpha}(G-U) - Z^\varepsilon_\varepsilon\|_\varepsilon + o(1).$$

As $T^\varepsilon_{\overline{m},\alpha}(G-U) = \varepsilon^{\overline{m}}T_{\overline{m},\alpha}(G-U)$, using the definition of the polynomial Z_ε we have

$$\|E_\varepsilon(F)\|_\varepsilon \leq \|T_{\overline{m},\alpha}(G-U) - V\|_\varepsilon + o(1) \leq \|T_{\overline{m},\alpha}(G-U) - V\|_0 + o(1),$$

as $\varepsilon \to 0$, and the right inequality of (11) holds.

To prove left inequality in (11)let (ε_k) be sequence tending to zero such that

$$\|E_{\varepsilon_j}(F)\|_{\varepsilon_j} \longrightarrow \underline{\lim}_{\varepsilon\to 0}\, \|E_\varepsilon(F)\|_\varepsilon.$$

By Theorem 4.6 we select a subsequence $\varepsilon_j \to 0$ in such a way that the following limits exist:

$$\lim_{j\to\infty} U_{\varepsilon_j} = \tilde{U}_0, \quad \tilde{U}_0 \in \mathcal{A}_0.$$

$$\lim_{j\to\infty} V_{\varepsilon_j} = \tilde{V}_0, \quad \tilde{V}_0 \in \Pi_k^{m,\alpha}.$$

Thus by (10) we have

$$\begin{aligned}\underline{\lim}_{\varepsilon\to 0}\|E_\varepsilon(F)\|_\varepsilon &= \|\Phi_{s,\alpha} - T_{\overline{m},\alpha}(\tilde{U}_0) - \tilde{V}_0\|_0 \\ &\geq \inf_{U\in\mathcal{A}_0,\ V\in\Pi_k^{m,\alpha}} \|T_{\overline{m},\alpha}(G-U) - V\|_0.\end{aligned}$$

□

Proposition 4.8. *Let $\|\ .\ \|_0$ be a strictly convex norm and assume that the subspace $\mathcal{A}$ fulfills Condition 4.5. Then there exists a unique solution $(U,V)\in\mathcal{A}_0\times\Pi_k^m$ to the minimum problem in Theorem 4.7 .*

Proof. If (U_1,V_1) and (U_2,V_2) are solutions to the minimum problem in Theorem 4.7, we have $\|T_{\overline{m},\alpha}(U_1)+V_1\|_0 = \|T_{\overline{m},\alpha}(U_2)+V_2\|_0$ with $(U_i,V_i)\in\mathcal{A}_0\times\Pi_k^{m,\alpha}$, $i=1,2$. Since $\|\ .\ \|_0$ is a strictly convex norm $T_{\overline{m},\alpha}(U_1+V_1) = T_{\overline{m},\alpha}(U_2+V_2)$, then by Condition 4.5, $U_1+V_1 = U_2+V_2$, but $\mathcal{A} = \mathcal{A}_0 \bigoplus \Pi_k^{m,\alpha}$. □

Theorem 4.9. *Let F be in $t^{\overline{m},\alpha}$, assume Condition 4.5 for $\mathcal{A}$, and that the minimum problem in 4.5 has a unique solution $(U_0,V_0)\in\mathcal{A}_0\times\Pi_k^{m,\alpha}$. Then $U_\varepsilon\to U_0$ and $V_\varepsilon\to V_0$ as $\varepsilon\to 0$. Moreover we have*

$$\|E_\varepsilon(F) - (T_{\overline{m},\alpha}(G-U_0) - V_0)\|_\varepsilon \longrightarrow 0.$$

Proof. By Theorem 4.6, Theorem 4.7 and (10) any convergent subsequence of the net $\{(U_\varepsilon,V_\varepsilon)\}_\varepsilon$ will converge to a solution of the minimum problem in Theorem 4.7. Thus if this solution is unique, the whole net converges to the solution. □

5. The limit of best approximation polynomials.

The main goal of this section will be to study the limit of $P_{\mathcal{A},\varepsilon,\alpha}(F)$ as $\varepsilon\to 0$. If $F\in t^{\overline{m},\alpha}$ and $G = F - T_{m,\alpha}(F)$ it will be enough to consider $P_{\mathcal{A},\varepsilon,\alpha}(G)$ as $\varepsilon\to 0$, since $P_{\mathcal{A},\varepsilon,\alpha}(F) = P_{\mathcal{A},\varepsilon,\alpha}(G) + T_{m,\alpha}(F)$.

We set as before $P_{\varepsilon,\alpha}(G) = P_{\mathcal{A},\varepsilon,\alpha}(G) = P_{\varepsilon,\alpha}(G) - T_{m,\alpha}(P_{\varepsilon,\alpha}(G)) + T_{m,\alpha}(P_{\varepsilon,\alpha}(G)) = U_\varepsilon + T_{m,\alpha}(P_{\varepsilon,\alpha}(G))$. Let F be in $t^{\overline{m},\alpha}$, then $\|\partial^\gamma P_{\varepsilon,\alpha}(0)\| \leq O(\varepsilon^{\overline{m}-\alpha.\gamma})$, for $\alpha.\gamma\leq\overline{m}$; see the proof of Theorem 4.6. Then $T_{m,\alpha}(P_{\varepsilon,\alpha}(G)) \longrightarrow 0$, $\varepsilon\to 0$. Thus $\lim_{\varepsilon\to 0} P_{\varepsilon,\alpha}(G) = \lim_{\varepsilon\to 0} U_\varepsilon = U_0$. From Theorem 4.7 and Theorem 4.8 , this polynomial exists whenever $\|\ .\ \|_0$ is a strictly convex norm. Then $\lim_{\varepsilon\to 0} P_{\mathcal{A},\varepsilon,\alpha}(F) = T_{m,\alpha}(F) + \lim_{\varepsilon\to 0} U_\varepsilon = T_{m,\alpha}(F) + U_0$, where U_0 together with V_0 are the unique solution to the minimizing problem

$$\min_{U\in\mathcal{A}_0,\ V\in\Pi_k^{m,\alpha}} \|T_{\overline{m},\alpha}(G-U) - V\|_0.$$

Thus if we set $P_0 = T_{m,\alpha}(F) + U_0 \in \mathcal{A}$ for $\lim_{\varepsilon\to 0} P_{\varepsilon,\alpha}(F)$ then P_0, in $\mathcal{A}_F = \mathcal{A}_0 + T_{m,\alpha}(F)$, will be the unique solution to the problem

$$\inf_{P\in\mathcal{A}_F,\ V\in\Pi_k^{m,\alpha}} \|T_{\overline{m},\alpha}(F-P) - V\|_0. \tag{12}$$

Thus, we have proved the following theorem.

Theorem 5.1. *Let F be in $t^{\overline{m},\alpha}$ and assume Condition 4.5 for $\mathcal{A}$, and that the minimum problem in (12) has a unique solution $(P_0, V_0) \in \mathcal{A}_F \times \Pi_k^{m,\alpha}$, and denote by $P_{\mathcal{A},\varepsilon,\alpha}(F)$ a polynomial in $\mathcal{A}$ which minimizes $\|F-P\|_\varepsilon^* = \|F^\varepsilon - P^\varepsilon\|_\varepsilon$, with $P \in \mathcal{A}$. Then $P_{\mathcal{A},\varepsilon,\alpha}(F) \to P_0$, as $\varepsilon \to 0$.*

6. On the best local approximation using Luxemburg norm.

We denote by μ_ε the measures given by (4), and let φ be a convex function such that $\varphi(0) = 0$, $\varphi(x) > 0$ if $x > 0$. For any measurable $F : B \subset \mathbb{R}^n \to \mathbb{R}^k$, set

$$\|F\|_\varepsilon = \inf\{\lambda > 0 : \int_B \sum_{i=1}^k \varphi\left(\frac{|f_i(t)|}{\lambda}\right) d\mu_\varepsilon(t) \le 1\}, \tag{13}$$

where $F(t) = (f_1(t), ..., f_k(t))$.

By Proposition (2.3) in [11] we have $\|F\|_\varepsilon$ converges to $\|F\|_0$, for any $F \in C_k(B)$. Moreover the family $\|F\|_\varepsilon, 0 \le \varepsilon \le 1$ has the properties (1),(2) y (3) of the section 2.

Recall that $\|F\|_0$ is the Luxemburg norm defined by (13) with the particular measure μ_0 defined by (6) and denote by $L_0^\varphi(B)$ the Orlicz Space equipped with the norm $\|F\|_0$. The following result is known .

Remark 6.1. *Let φ be a strictly convex function, then $L_0^\varphi(B)$ is a strictly convex Banach space with the Luxemburg norm $\|.\|_0$.*

By Remak 6.1 we can use Proposition 4.8 and Theorem 5.1 for the Luxemburg norm $\|.\|_0$ when φ is a strictly convex function. Also we are free to apply Theorem 5.1 in [11]. We apply these results in the particular situation described below.

Given $f : [-1,1] \to \mathbb{R}$ and $-1 < x_1 < ... < x_k < 1$, set $F(t) = (f(x_1 + t), ..., f(x_k + t))$ and the norm $\|F\|_\varepsilon = \|F\|$ as in (13) and the measure $d\mu_\varepsilon$ is the Lebesgue measure dt.

Theorem 6.2. *Let φ be a strictly convex function and let $P_\varepsilon \in \pi^m$ be the unique solution of the minimum problem*

$$\|f-P\|_\varepsilon^* = inf\{\lambda > 0 : \Sigma_{i=1}^k \int_{x_i-\varepsilon}^{x_i+\varepsilon} \varphi\left(\frac{|f(x)-P(x)|}{\lambda}\right)\frac{dx}{\varepsilon} \le 1\},$$

where $P \in \pi^m$. Then for a smooth function f, P_ε converges to a polynomial $P_0 \in \pi^m$, which is uniquely determined by the solution of the minimum problem in (4) of [11].

Now we will assume more restrictive conditions on the strictly convex function φ, namely $\lim_{x\to 0} \frac{\varphi(x)}{x} = 0$, $\lim_{x\to\infty} \frac{\varphi(x)}{x} = \infty$ and $\widehat{\varphi}(\lambda) = \lim_{x\to\infty} \frac{\varphi(\lambda x)}{x}$ exists and it is a finite number for every $\lambda \ge 0$. Clearly $\widehat{\varphi}$ is convex function, $\widehat{\varphi}(0) = 0$ and it is easy to see that $\widehat{\varphi}(x) = x^p$, for $x \ge 0$, and if $\widehat{\varphi}(2) > 2$, we have $1 < p < \infty$ see [13]. From now on assume all the above conditions on the function φ.

Theorem 6.3. *For any $\varepsilon > 0$, let $Q_\varepsilon(f)$ be the unique polynomial in π^m which minimizes*

$$\|(f-P)\mathcal{X}_{V_\varepsilon}\|_0,$$

$P \in \pi^m$ and $V_\varepsilon = \bigcup_{i=1}^k (x_i - \varepsilon, x_i + \varepsilon)$. Then the limit $Q_0(f) = \lim_{\varepsilon\to 0} Q_\varepsilon(f)$ exist for smooth functions f.

Theorem 6.3 may be obtained using results of [13] and [20]. For the case $m+1 < k$, the polynomial $Q_0(f)$ it is very easy to characterize as the unique element $Q_0(f) \in \pi^m$ which minimizes the problem

$$\sum_{i=1}^k |(f(x_i) - Q(x_i))|^p,$$

$Q \in \pi^m$, see [13]. For the case $m+1 = kq + r$, $r > 0$ also $Q_0(f)$ can be obtained as a discrete minimum L^p problem as in [20].

The best local approximation polynomials $P_0(f)$ described in Theorem 6.2 and $Q_0(f)$ in Theorem 6.3 are different polynomials. Indeed, it is rather straightforward to obtain the next result when f is a continuous function at each point $x_1, ..., x_k$ and φ just a strictly convex function $\varphi(0) = 0$.

Theorem 6.4. *For $m+1 < k$, and $\varepsilon > 0$ let P_ε the unique polynomial which minimizes*

$$inf\{\lambda > 0 : \Sigma_{i=1}^k \int_{x_i-\varepsilon}^{x_i+\varepsilon} \varphi\left(\frac{|f(x)-P(x)|}{\lambda}\right)\frac{dx}{\varepsilon} \le 1\},$$

$P \in \pi^m$. Then the limit $P_0(f) = \lim_{\varepsilon\to 0} P_\varepsilon(f)$ exist and it is characterized as the unique $Q_0 \in \pi^m$, which minimizes

$$\inf\{\lambda > 0 : \Sigma_{i=1}^k 2\varphi\left(\frac{|f(x_i)-Q(x_i)|}{\lambda}\right) \le 1\},$$

$Q \in \pi^m$.

We point out that to prove the existence of the polynomial $Q_0(f)$ in Theorem 6.3 still remains an open problem when φ is just a strictly convex function and the existence of the function $\widehat{\varphi}$ is not required.

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Norma Yanzón
Instituto de Matemática Aplicada San Luis. CONICET and
Departamento de Matemática,
Universidad Nacional de San Luis,
(5700) San Luis, Argentina
nbyanzon@unsl.edu.ar

Felipe Zó
Instituto de Matemática Aplicada San Luis. CONICET and
Departamento de Matemática,
Universidad Nacional de San Luis,
(5700) San Luis, Argentina
fzo@unsl.edu.ar

Recibido: 1 de septiembrede 2008
Aceptado: 25 de noviembre de 2008